Lecture Notes
on
Dynamic Meteorology

For

Integrated Meteorological Training Course

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Course content

- **Equation of Motion:** Frames of reference, Vector equation of motion in inertial & non-inertial frame (No derivation). Local tangential coordinate system. Equation of motion (in component form), explanation (without derivation) of all the terms. Pressure as a vertical co-ordinate & its usefulness. Horizontal equation of motion with pressure as a vertical co-ordinate. Atmospheric forces: Real & apparent forces, body & surface forces: Coriolis force, Pressure gradient force, Centrifugal force, Gravity and Gravitation.
- **Geostrophic approximation:** Definition and properties of geostrophic wind. Vectorial expression for geostrophic wind. Schematic diagram to show how geostrophic balance can be achieved. Ageostrophic wind: Definition, vectorial expression and its property.
- **Hydrostatic approximation:** Hydrostatic equation. What is hydrostatic approximation? Discussion on the validity of this approximation. Using above approximation, definition of atmospheric pressure at any point. Definition of geopotential and geopotential height of a point and corresponding units. Hypsometric equation (no derivation) and its use in computing thickness of a layer of atmosphere
- **Balanced flow:** Introduction to natural co-ordinate system. Horizontal equation of motion in natural co-ordinate. Gradient balance and gradient wind. Physically possible different gradient flow. Examples. Limits for gradient flow. Special cases of gradient balance:- geostrophic balance, inertial balance, and cyclostrophic balance. Examples.
- **Vertical variation of wind:** Concept of vertical wind shear. Schematic explanation for horizontal temperature gradient leading to vertical shear of geostrophic wind. Thermal wind: Definition, Thermal wind equation and properties of thermal wind. Application of the concept of thermal wind: cold and warm advection associated with veering/backing of geostrophic wind, Jet stream, cold/warm core lows/high. Concept of barotropic and baroclinic atmosphere.
- **Kinematics of wind and pressure field:** Definition of Streamlines and trajectory, relation between them, streamline patterns for pure translations, pure divergence, pure rotations and deformations. Definition and mathematical expression for center of Lows/ highs, equation for trough/ ridge and Col.
- **Conservation of mass:** Equation of continuity, Dines compensation principle, Concept of the level of non-divergence. Moisture continuity equation.
- **Divergence & vorticity:** Definition of Divergence and vorticity & their mathematical expression. Illustration by typical cases on synoptic charts.
- **Introduction to PBL:** Definition of PBL, Importance of PBL, Convective turbulence & mechanical turbulence, depth of PBL, Static stability, Richardson number. Different sub layers in PBL.
- **Practical Dynamic Met. :** Computation of horizontal divergence & vorticity at a point on the streamline using curvature method. Computation of the above and vertical velocity using finite difference grid, Computation of precipitable water content, Computation of geostrophic wind, thermal wind, thermal advection, moisture flux and vertical wind.

Chapter-1

Equation of Motion

To discuss about equation of motion, one first should know about reference frame, because equation of motion is always written/talked about, with respect to some frame of reference.

Thus, we start this subject from the concept of reference frame.

Reference frame:

It is defined as a system consisting of some fixed points and lines, which are just sufficient to locate a point in space uniquely.

Reference frames are categorized into two types, viz., the inertial and non –inertial reference frames.

Reference frames

Inertial **Non –Inertial**

Inertial reference frame: It is that reference frame which is either fixed or moves with uniform velocity with respect to fixed star. It is also known as non- accelerated reference frame or absolute frame of reference. Newton's laws of motion can be applied in this reference frame without adding any additional force.

Non-inertial reference frame: This reference frame moves with some acceleration with respect to fixed star. This is also known as accelerated reference frame. Due to the acceleration of the reference frame, an object placed on it experiences an extra forcing. Thus this extra forcing has to be added to apply the Newton's laws of motion in this reference frame.

Governing Equations:- The atmospheric motions are governed by four conservation laws. They are respectively the

- Conservation of momentum
- Conservation of mass
- Conservation of moisture
- Conservation of energy

The above conservation laws are expressed by six differential equations. These six differential equations and the equation of state are known as governing equations.

Now we shall derive the governing equation:

<u>Equation of motion</u>:

This is simply the mathematical statement of the Newtons 2nd law of motion. According to this law we have, Acceleration = Vector sum of forces per unit mass.

Now if $\vec{V}_a$ is the velocity vector in inertial frame of reference, then equation motion in inertial frame can be written as :

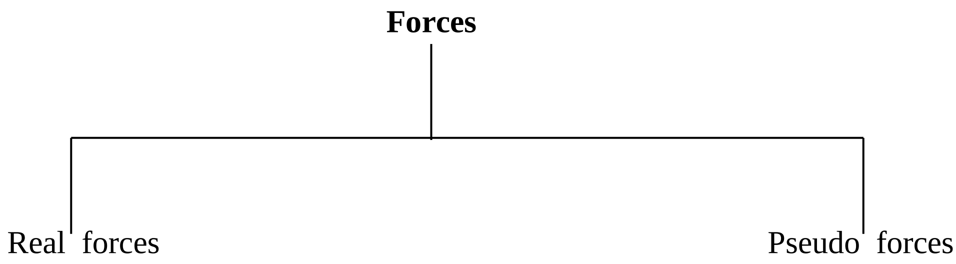
$\frac{d_a V_a}{dt}$ = Vector sum of real forces, where the symbol $\frac{d_a}{dt}$ represents the time derivative in an inertial frame of reference.

For the atmospheric motions, the real forces are the pressure gradient force, gravitational force and the fractional (Viscous) force.

In this context, it is worth to mention that the forces may also be categorized into different categories.

Again this categorization may also be done in different ways. One categorization of forces is based on their existence in different reference frame and other one is based on the direction of their line of action.

<u>Categorization I</u> :-



Real forces are those, which exist in all reference frames. For example gravitational force, viscous force, electro magnetic forces are all real forces. On the other hand Pseudo forces exist in a non-inertial reference frame only. For example, centrifugal force, Coriolis force are Pseudo forces, as they exist in a rotational frame only.

Categorization-II:

Forces

Body force | Surface force

Surface forces are those, the line of action of which is normal to the surface of the object and depend on the area of the surface across which it act but not on the mass of the object. Pressure gradient force, Viscous force are examples of surface force. On the other hand body force acts at the centre of mass of the object and depends on mass of the object but not on the surface area of the object. Examples of such forces are Coriolis force, Gravitational force, Centrifugal force.

<u>Illustration about pressure gradient force:-</u>

This force arises from the spatial gradient of pressure. To understand how gradient of pressure results into a net forcing, we consider a cylinder with ends A & B, as shown in the adjoining figure.

Let the pressure exerted on face A is P_A and that on face B is P_B, if S is the cross sectional area of the cylinder, then the force exerted at face A=P_AS and it is directed in the direction from A to B. Similarly force exerted at face B is P_BS and it is directed from B to A.

Thus, there is a resultant force along the direction from A to B, which is equal to $P_AS-P_BS=(P_A-P_B)S$.

Clearly, this resultant force $\neq 0$ iff $P_A \neq P_B$. Thus the resultant force is solely due to the gradient of pressure.

This illustrates how the gradient of pressure results into a forcing. This force is known as pressure gradient force. It can be shown that the expression for pressure gradient force at an arbitrary point $(x_0, y_0, z_0) = -\frac{1}{\rho}\left(\vec{\nabla}P\right)_{(x_0,y_0,z_0)}$.

Clearly PGF $\neq 0$ if and only if $\left|\vec{\nabla}p\right| \neq 0$. As gradient of any scalar is directed from its low value to its high value, hence pressure gradient is directed from low to high-pressure area and due to the presence of a minus sign, it follows that PGF force is directed from high to low pressure. This force is always normal to the isobars.

Viscous force:- We know that in a sheared fluid flow, there is a relative motion at the interface between two adjacent fluid layers. This relative motion causes a drag on the motion of a fluid layer exerted by other one. This drag is proportional to the shear of the fluid velocity. This is known as viscous stress.

Thus, if u, v, w are the components of wind, then the different components of viscous stress are given by the following stress matrix

	X	Y	Z
u	τ_{xx}	τ_{yx}	τ_{zx}
v	τ_{xy}	τ_{yy}	τ_{zy}
w	τ_{xz}	τ_{yz}	τ_{zz}

Clearly viscous stress is a tensor, where, $\tau_{ij} = \mu \frac{\partial u_j}{\partial x_i}$ and $i, j = 1,2,3$, $x_1 = x, x_2 = y, x_3 = z, u_1 = u, u_2 = v, u_3 = w$ and μ is the dynamic coefficient of friction.

It can be shown that the net viscous force along the positive x-direction, y direction and z-direction are respectively $\nu\nabla^2 u$, $\nu\nabla^2 v$ and $\nu\nabla^2 w$. Thus the frictional fore (Viscous force) in vector form is given by, $\vec{F} = \hat{i}\,\nu\nabla^2 u + \hat{j}\,\nu\nabla^2 v + \hat{k}\,\nu\nabla^2 w$.

Equation of motion in a rotating reference frame:

It can be shown that, for an arbitrary vector $\vec{B}$, time rate of change of $\vec{B}$ with respect to absolute reference frame (Ox,Oy,Oz) = Time rate of change of $\vec{B}$ with respect to rotating reference frame (Ox′,Oy′,Oz′)+ Time rate of change of $\vec{B}$ due to rotation of the reference frame (Ox′,Oy′,Oz′) with respect to (Ox,Oy,Oz). Again it can be shown that the rate of change of $\vec{B}$ due to rotation of the reference frame (Ox′,Oy′,Oz′) with respect to (Ox,Oy,Oz) is $\vec{\Omega}\times\vec{B}$.

Hence, $\dfrac{d_a\vec{B}}{dt} = \dfrac{d\vec{B}}{dt} + \vec{\Omega}\times\vec{B}$.

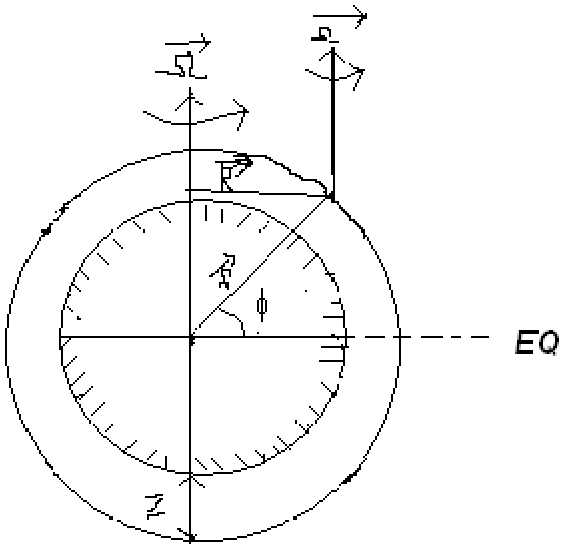
To obtain the equation of motion in rotating reference frame, in the above expression we substitute $\vec{B}$ by $\vec{r}$ to obtain $\vec{V}_a$ and then by $\vec{V}_a$.

So, $\vec{V}_a = \dfrac{d_a\vec{r}}{dt} = \dfrac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r} = \vec{V} + \vec{\Omega}\times\vec{r}$

And $\dfrac{d_a\vec{V}_a}{dt} = \dfrac{d}{dt}(\vec{V} + \vec{\Omega}\times\vec{r}) + \vec{\Omega}\times(\vec{V} + \vec{\Omega}\times\vec{r}) = \dfrac{d\vec{V}}{dt} + 2(\vec{\Omega}\times\vec{V}) + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$

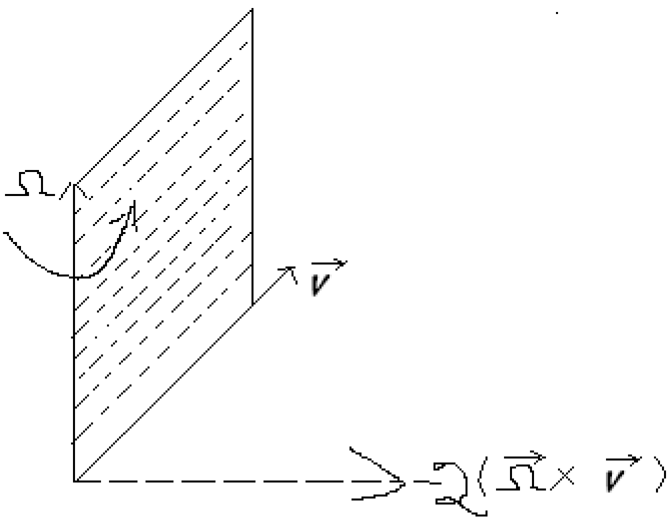
To find out $\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$ the adjoining figure may be referred to, where a meridional cross section of earth passing through an object at latitude ϕ has been shown. In this figure radius of the ϕ-latitude circle is $\left|\vec{R}\right| = \left|\vec{r}\right|\cos\phi$. Now, $\left|\vec{\Omega}\times\vec{r}\right| = \left|\vec{\Omega}\right|\left|\vec{r}\right|\sin(90^0 - \phi) = \left|\vec{\Omega}\right|\left|\vec{R}\right|\sin 90^0 = \left|\vec{\Omega}x\vec{R}\right|$. Since the vectors $\vec{\Omega}, \vec{r}, \vec{R}$ are coplanar, hence any vector normal to the plane containing $\vec{\Omega}$ and $\vec{r}$, will be so to the plane containing $\vec{\Omega}$ and $\vec{R}$ also.

Hence, $\vec{\Omega}x\vec{r} = \vec{\Omega}x\vec{R}$. Hence, $\vec{\Omega}x\left(\vec{\Omega}x\vec{R}\right) = -\Omega^2\vec{R}$.



Coriolis force :- The coriolis force is given by

$$CoF = -2(\Omega \times V)$$

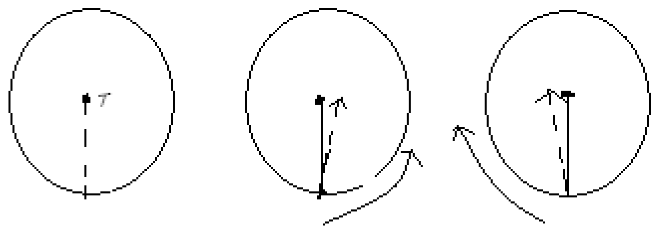


It is clear from the above expression that CoF is perpendicular to the velocity vector V of the object as well as to the angular velocity vector Ω of earth. So, CoF is directed along the direction of movement of right-handed screw as it is rotated from V to Ω. This CoF is to the right of the direction of V i.e., right of the direction of movement, in the northern hemisphere. Its effect is to deflect one moving object to the right of it's direction of movement (in the N.H.). Hence it is also called a deflecting force.

As this force is perpendicular to the direction of movement, hence this force does not do any work. One important thing to be noted that CoF comes in to play for a moving object and in a rotational frame of reference ($\because \Omega$ x V$\neq 0$ iff $\Omega \neq$ O & V$\neq$ O).

It is the rotation of the reference frame only, which is responsible for deflecting a moving object to illustrate it following example may be referred.

Diagram:-



In the adjoining figure one circular disc has been shown at the center of which one target has been kept from a point on the rim a bullet is fired. In the first case the disc is stationary and the bullet moves along a straight path and hits the target. In the second case the disc is rotating in an anticlockwise sense ($\Omega \geq$ O). As it is seen instead of following a straight-line path the bullet has been deflected to the right of it's direction of movement and fells to hit the target. In the third case as the disc rotates in a clockwise sense, the bullet has been deflected to the left of its direction of movement.

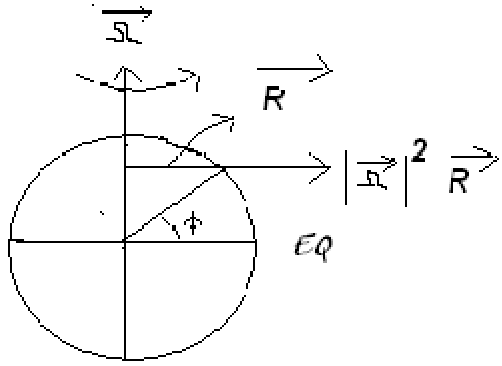
Thus, rotation of the disc is solely responsible for the deflection of the moving bullet from its direction of movement.

As the existence this force determined by the rotation/non- rotation of a reference frame, it is a pseudo force.

Clearly at the pole $\vec{\Omega}$ is parallel to $\hat{k}$ (in N.H.) , hence at the pole CoF is horizontal, because if $\vec{\Omega} = |\Omega|\hat{k}$ and $\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$, then $\vec{\Omega}\ X\,\vec{V} = |\Omega|[-\hat{i}v + \hat{j}u]$.

Similarly at the equator, Ω is parallel to ĵ, hence coriolis force will be in the x-z plane, i.e. in a vertical plane. If the vertical component of velocity is very small compared to the horizontal component, then CoF will be approximately vertical, because, $\vec{\Omega}\ X\,\vec{V} = |\Omega|[\hat{i}w - \hat{k}u]$ and if |w| < < |u|, then $\vec{\Omega}\ X\,\vec{V} = -|\Omega|u\hat{k}$. Hence Coriolis force at the equator will be approximately vertical.

Centrifugal force:- Vectorially this force is expressed as $\Omega^2\vec{R}$, Where R is perpendicular to the earths axis and is directed away from it, as shown in the adjoining figure.



Like CoF this force also exists in a rotating frame of reference. So this is also the pseudo force.

It is well known that when an object executes a curvilinear motion (for example circular motion) it is always subjected to a force directed towards the center of the curvature. If the above motion viewed from an absolute reference frame, then the object will appear to change its position at every instance. Now if the same is viewed from a rotating frame which is rotating with same angular velocity as that of the object, then the object will appear to be stationary with respect to the rotating frame. But already, the object is subjected to a real force viz the radial or centripetal force directed towards the center. Hence for the object to appear as stationary with respect to rotating frame, it must be subjected to an equal and opposite force. This is nothing but centrifugal force.

Existence of this force felt by the outward deflection, instead of tangential deflection of the object when it suddenly stops rotation.

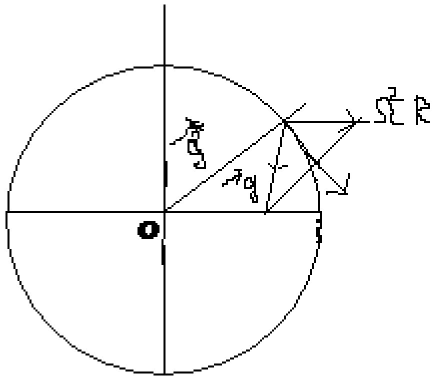
<u>Effective gravity or gravity and a plausible cause for flattening of earth surface at pole</u>:

Gravitational attraction is defined following universal law of gravitation. Accordingly, the gravitational force per unit mass, denoted by $\vec{g}^*$ is given by, $\vec{g}^* = -\frac{GM}{r^2}\left(\frac{\vec{r}}{r}\right)$.

Effective gravity or simply gravity, denoted by $\vec{g}$, is define as the resultant of centrifugal force $(\Omega^2\vec{R})$ and the gravitational force $\vec{g}^*$

Thus

$$\vec{g} = \vec{g}^{*} + (\Omega^2 \vec{R})$$



To offer a plausible explanation for the flattening of earth surface at pole, we may consider that earths shape was that of a perfect sphere initially. Now, $\vec{g}^{*}$ is directed radially towards the center of the earth, so that it does not have a tangential component towards equator. But, $\vec{g}$ is not radially directed, so it must have a component, tangential to spherical earths surface, towards equator. From simple geometry it will follow that there is no other force balancing this equator ward component of $\vec{g}$. Hence masses placed on earth surface are exerted upon by this net equator ward force, which caused the masses to move towards equator along the earth surface. Now, this movement of masses towards equator along the earths surface would continue till a state of equilibrium has been reached. This might have caused a bulging of earth near equator.

As in the present state no mass is acted upon by any unbalanced equator ward force along earth surface. It may be assumed that state of equilibrium has been reached. Now clearly in this state, $\vec{g}$ does not have any equator ward component, which requires that at every point on the earths surface $\vec{g}$ must be normal to earth surfaces, which is not possible as long as a perfect spherical earth is considered. Hence shape of earth is not perfect sphere but an oblate spheroid with bulging near equator. Gravitational force $\vec{g}^{*}$ is always directed towards center, where as $\vec{g}$ is so only at equator & at pole.

Gravity $\vec{g}$ is always normal to the earths surface, where as gravitational force $\vec{g}^{*}$ is not necessary so at each point on the earths surface.

<u>Equation of motion in tangential local co-ordinate:-</u>

What is tangential local co-ordinate?

To understand tangential local co-ordinate, we consider an object placed at some point on the earth’s surface. Then a plane, tangent to the earth’s surface at that point is considered. On this tangential plane through the above point two mutually perpendicular axes, one pointing towards east other towards north is considered.

The rectangular co-ordinate system, consisting of the above mention two axes as horizontal and another axis pointing through the point perpendicular to the tangent plane as vertical is known as tangential local co-ordinate system.

The unit vectors $\hat{i}, \hat{j}$ *and* $\hat{k}$ are considered pointing towards east, north and local vertical.

With respect to the above cartesian frame of reference, we have,

$$\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$$

$$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{g} = -g\hat{k}$$

and $\quad \vec{F} = \eta\left(\hat{i}\nabla^2 u + \hat{j}\nabla^2 v + \hat{k}\nabla^2 w\right).$

The Coriolis force in this co-ordinate system, at latitude 'ϕ' is given by

COF = $-2\Omega\left[\hat{i}(w\cos\phi - v\sin\phi) + \hat{j}(u\sin\phi) - \hat{k}(u\cos\phi)\right].$

Hence the component wise form of the equation of motion in tangential local co-ordinate system can be written as :

$$\frac{du}{dt} = -\alpha\frac{\partial p}{\partial x} + 2\Omega\,(\mathrm{v}\mathrm{Sin}\,\phi - \mathrm{w}\mathrm{Cos}\,\phi) + \eta\nabla^2\mathrm{u}$$

$$\frac{dv}{dt} = -\alpha\frac{\partial p}{\partial y} - 2\Omega uSin\phi + \eta\nabla^2\mathrm{v}$$

$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} - \mathrm{g} + 2\Omega u\cos\phi + \eta\nabla^2\mathrm{w}$$

Total change, In-situ change and advectional change:-

Let us consider an arbitrary function $f(x,y,z,t)$ of space (x,y,z) and time (t). Now a change δx in x, δy in y , δz in z and δt in t, result in the following changes of f respectively:

$$\frac{\partial f}{\partial x}\delta x, \frac{\partial f}{\partial y}\delta y, \frac{\partial f}{\partial z}\delta z \quad and \quad \frac{\partial f}{\partial t}\delta t$$

Hence a total change in $f(x,y,z,t)$ due to simultaneous changes in $\delta x, \delta y, \delta z, \delta t$ in x, y, z and t is given by

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t$$

$$= \frac{\partial f}{\partial t}\delta t + \delta \vec{r}.\vec{\nabla} f$$

Hence $\frac{df}{dt} = \underset{\delta t \to 0}{Lt} \frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} + \underset{\delta t \to 0}{Lt} \frac{\delta \vec{r}}{\delta t}.\vec{\nabla} f = \frac{\partial f}{\partial t} + \vec{V}.\vec{\nabla} f$

Thus the total rate of change of 'f' with respect to't' consists of two parts, viz., $\frac{\partial f}{\partial t}$ and $\vec{V}.\vec{\nabla} f$. $\frac{\partial f}{\partial t}$ is known as in-situ rate of change, because this change does not involve any change in location, where as $\vec{V}.\vec{\nabla} f$ is the charge in f due to change in x,y & z i.e. due to change in position . This change is due to transport of f by wind.

Transport of any physical quantity by horizontal wind is known as advection and that by vertical wind is known as convection. Advection is said to be positive at a point if at that point

$-\vec{V}.\vec{\nabla} f > 0$, i.e. if $\vec{V}$ is directed to that point from higher value of f .

The derivative $\frac{\partial}{\partial t}$ is called **Eulerian derivative**, where as $\frac{d}{dt}$ is called **Lagrangian derivative**. Thus the complete form of the equation of motion in tangential local co-ordinate system can be written as ,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\alpha\frac{\partial p}{\partial x} + 2\Omega(vSin\phi - wCos\phi) + \eta\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\alpha\frac{\partial p}{\partial y} - 2\Omega uSin\phi + \eta\nabla^2 v .$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\alpha\frac{\partial p}{\partial z} + 2\Omega uCos\phi + \eta\nabla^2 w .$$

Change of vertical co-ordinate:

Till now we have dealt the governing equations, taking 'Z' as vertical co-ordinate. Here we shall see, what are the other parameters, which can be used as vertical co-ordinate.

An arbitrary quantity, say, 'ξ' may be used as a vertical co-ordinate if 'ξ' is a monotonic (increasing or decreasing) function of 'Z', i.e.; if ξ either steadily increases or decreases with 'Z'.

Mathematically $\frac{\partial \xi}{\partial Z}$ is either positive throughout in the vertical or negative in the vertical.

Considering the above condition, it can be seen that $\frac{\partial p}{\partial Z} = -g\rho$ < 0. Hence p is a monotonic decreasing function of Z; therefore pressure 'p' can be used as a vertical co-ordinate.

Similarly it can be seen that $\frac{\partial \theta}{\partial z} > 0$, so potential temperature 'θ' being a monotonic decreasing function of Z, may be used as a vertical co-ordinate.

Now we shall discuss the horizontal momentum equation in 'p' co-ordinate system. In this system horizontal planes are constant 'p' surfaces, i.e., isobaric surfaces.

It can be shown that the gradient of a scalar $\xi(x, y, z)$ with respect to an arbitrary vertical co-ordinate $\phi(x, y, z)$ can be expressed as :

$$\vec{\nabla}_\phi \xi = \vec{\nabla}_z \xi + \left(\frac{\partial \xi}{\partial z}\right) \vec{\nabla}_\phi Z.$$

Now we are in a position to have the expression for horizontal pressure gradient force in different vertical coordinate. We know horizontal pressure gradient force in Z- co-ordinate system is $-\frac{1}{\rho}\vec{\nabla}_z p$.

In the above expression, we put $\xi = p$ and $\phi = p$ and then we have

$$\vec{\nabla}_p p = \vec{\nabla}_z p + \frac{\partial p}{\partial z}\vec{\nabla}_p Z$$

$$\Rightarrow -\frac{1}{\rho}\vec{\nabla}_z p = -g\vec{\nabla}_p Z$$

Hence the horizontal pressure gradient force in p-co-ordinate can be expressed as, $-\vec{\nabla}_p (gz)$.

Chapter - 2

Geostrophic approximation

To understand geostrophic approximation, first we should understand some basic concepts of atmospheric scale analysis.

Before discussing about scale analysis, we should first understand what is order of magnitude of a physical parameter and what scale of atmospheric motion is.

Order of magnitude of physical quantity:- Suppose observed value of a physical quantity ,say " f " is γ. Then γ can be written as r x10^N, where r is a real number between 1 to 9 , N is any integer (positive/Negative).

Then order of magnitude of f $[O(f)]$ is determined as follows:

	N +ve	N-ve
r<5	$[O(f)] = 10^N$	$[O(f)] = 10^N$
r>5	$[O(f)] = 10^{N+1}$	$[O(f)] = 10^{N+1}$

For example, say observed value of surface pressure is 992hpa. Then it may be written 0.992x 10^3 . Here r = 0.992<5 and N = 3. So, order of magnitude of surface pressure is 10^3 hPa.

Scale of atmospheric motion:- It is the order of magnitude of maximum horizontal extent of some atmosphere motion.

CATEGORY	**SCALE OF MOTION**
Micro Scale	$\leq 10km$
Meso Scale	10-10^2 km
Synoptic Scale	10^3 km
Macro Scale	10^4 Km and above

<u>Scale analysis</u>:- Atmospheric motion is governed by some conservation laws which when are expressed as partial differential equations known as governing equations.

Each governing equation deals with tendency of some physical parameter and possible mechanisms for that tendency.

In the governing equation terms other than the tendency term, represent certain mechanism responsible for tendency. Now it is not necessary that all mechanisms will contribute equally towards the change of the some parameter, rather this contribution crucially depends on the scale of the motion under consideration.

Scale analysis is a convenient technique to estimate quantitatively the contribution of individual mechanism and then after comparison certain terms with comparatively less significant are ignored.

The following steps are to be performed to carry out scale analysis of any governing equation:-

- To determine typical order of magnitude of field variables for a particular type of motion at the given latitudinal belt.
- Using the above to find out the typical order of magnitude of individual term of the governing equation.
- To retain those terms with highest order of magnitude, discarding others.

Performing scale analysis of equation of horizontal motion for mid-latitude synoptic scale system it can be shown that the order of magnitude of the terms representing pressure gradient force and Coriolis force is $10^{-3}ms^{-2}$ and the order of magnitude of all other terms are $10^{-4}ms^{-2}$ or less.

So, following the principle of scale analysis, these two terms may be retained discarding the other terms. Hence the above two equations for horizontal motion are simplified to

$$0 \approx -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

$$0 \approx -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu$$

The above two approximate equations are known as geostrophic approximation. The values of u,v having dimension of wind, obtained from the equations:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + fv = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu=0$$

are known as **geostrophic wind**, which is vectorially expressed as

$$\vec{V}_g=\frac{\hat{k}}{\rho f}x\vec{\nabla}_H p$$

Similarly, Geostrophic wind in p-coordinate is given as $\vec{V}_g=\frac{g}{f}\left(\hat{k}\,X\,\vec{\nabla}_p z\right)$

It may be noted that geostrophic wind can be obtained from the spatial distribution of the pressure field.

From the foregoing discussion it follows that geostrophic wind is horizontal, frictionless flow, results from a complete balance between PGF & CoF. This wind is proportional to the magnitude of pressure gradient and it is parallel to the isobars keeping low pressure to its left in the northern hemisphere (opposite in southern hemisphere).

Thus scale analysis of horizontal equation of motion leads to geostrophic approximation.

<u>Ageostrophic flow:</u>

Let (u,v) are the horizontal components of observed (actual) wind at a point and (u_g,v_g) are the geostrophic approximation of the above, obtained from the horizontal distribution of pressure, as

$$u_g=-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \& \qquad v_g=-\frac{1}{\rho f}\frac{\partial p}{\partial x}.$$

Ageostrophic wind at that point is defined as the difference between actual wind and geostrophic approximation. Of actual wind at that point. If the horizontal components of ageostrophic wind are denoted by U_a & V_a respectively, then

$U_a = U-U_g$ & $V_a = V-V_g$.

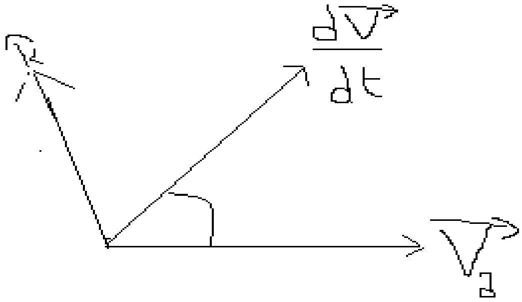
Horizontal equation of motion, neglecting the frictional effect, may be written as

$$\frac{dv}{dt}=-\frac{1}{\rho}\nabla p+fkX\vec{V}$$

Again $\quad 0 \approx -\frac{1}{\rho}\nabla p + fkX\vec{V}_g$

Hence $\quad \frac{d\vec{v}}{dt} = fkX\vec{V}_a$

⇨ $\quad V_a = \frac{1}{f}(\frac{d\vec{v}}{dt}Xk)$



Thus we have :

- Ageostrophic wind is proportional to the magnitude of acceleration.
- It is perpendicular to the acceleration.
- It is to the right of acceleration.

So, if there is an imbalance between the PGF &CoF, which causes the motion to be accelerated, then ageostrophic wind will be towards PGF or CoF according as which one is more.

Chapter-3
Hydrostatic approximation

Hydrostatic balance refers to the balance between the vertical component of pressure gradient force and the gravity. In the last chapter we have seen that by performing scale analysis, for synoptic scale mid latitude system, the horizontal momentum equation simplified to Geostrophic approximation. Similarly the vertical momentum equation after scale analysis for synoptic scale system simplified to hydrostatic approximation. According to this approximation, the vertical component of pressure gradient force and the gravity are approximately in balance for synoptic or larger scale system. Mathematically it can be expressed as:

$0 \approx -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$, Which is known as hydrostatic approximation**.**

This approximation is valid till there is no net vertical acceleration. In case of smaller scale motion, viz., thunder storm, tornado etc, vertical component of pressure gradient force may exceed gravity significantly, resulting into net vertical acceleration. For such situations, hydrostatic approximation is not valid.

Some corollaries from hydrostatic approximation :

Definition of atmospheric pressure:

From the hydrostatic approximation we have,

$$\frac{\partial p}{\partial z} \approx -g\rho$$

Now integrating the above equation from an arbitrary pressure level, say ‘p’, to the top of the atmosphere, we obtain

$$\int_p^0 \frac{\partial p}{\partial z} dz \approx \int_z^\infty -g\rho dz$$

=> $$\mathrm{P} \approx \int_z^\infty g\rho dz$$

Hypsometric equation:

If the hydrostatic approximation is integrated between two pressure levels, say $Z=Z_L$ &$Z=Z_U$, with pressure, say $P=P_L$ & $P=P_U$, we obtain

$$Z_u - Z_L \approx -\int_{P_L}^{P_U} \frac{\partial p}{g\rho} = \frac{R}{g}\int_{P_U}^{P_L} T d(\ln p)$$

$$\text{Thickness} = (Z_U - Z_L) \quad = \frac{R}{g} < T > \ln(\frac{P_L}{P_U}),$$

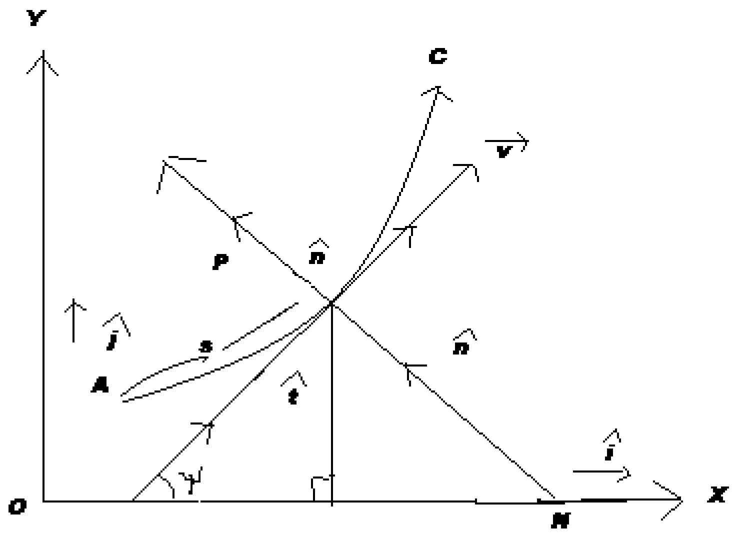
Where <T> is the mean temperature of air in the layer between $Z=Z_L$ & $Z=Z_u$, given by

$$<T> = \frac{\int_{P_U}^{P_L} T d(\ln p)}{\int_{P_U}^{P_L} d(\ln p)}$$

This is known as **hypsometric equation. This method of computation of thickness is referred to as Isothermal method of computation of thickness.**

Chapter-4
Balanced flow

Here we shall introduce another one co-ordinate system, which moves along with flow. This co-ordinate system is known as Natural co-ordinate system.
We have seen earlier that any co-ordinate system is defined by the unit vectors along the co-ordinate axis.



To define the natural co-ordinate, we consider the flow along the directed path as shown in the adjoining figure. Natural co-ordinate at an arbitrary point 'P' on the flow is defined by the co-ordinate axes, one along the direction of wind at that point, another perpendicular to it and to it's left and last one is perpendicular to both.
Three unit vectors $\hat{t}, \hat{n}, \hat{k}$ are defined along these three co-ordinate axis, they form a triad, i.e.,

$$\hat{t} \times \hat{n} = \hat{k}, \hat{n} \times \hat{k} = \hat{t}, \hat{k} \times \hat{t} = \hat{n}.$$

$\hat{t}$ is known as unit tangent vector as it is in the direction of tangent at P. It can be shown that

$\vec{v} = v\hat{t}$; Where v is the speed of wind.

Hence in natural co-ordinate wind velocity is always tangential and positive definite, i.e. existence of wind itself ensures that it is positive.
From the adjoining figure using simple geometry it can be shown that

$$\hat{t} = \hat{i}\cos\psi + \hat{j}\sin\psi$$

and $$\hat{n} = -\hat{i}\sin\psi + \hat{j}\cos\psi$$

Now it can be shown that $\frac{d\vec{v}}{dt} = \hat{t}\frac{dv}{dt} + V^2 k.\hat{n}$, Where k is the curvature of the path along which flow takes place.
Thus we see that, although velocity is solely tangential, acceleration has a normal component, which is known as radial or centripetal or normal acceleration. It is due mainly to the curvature of flow and is always towards the center of curvature.

Hence tangential acceleration $=\frac{dv}{dt}$, and normal acceleration = $V^2 k = \frac{V^2}{R}$, where 'R' is the radius of curvature.
As the coriolis force is always normal to the direction of flow and to the right of the flow, hence it follows that the horizontal component of coriolis force can be given by $-fv\hat{n}$.

it can also be shown that, $-\frac{1}{\rho}\nabla_H p = -\frac{1}{\rho}(\hat{t}\frac{\partial p}{\partial s} + \hat{n}\frac{\partial p}{\partial n})$

Hence, the equation of motion for horizontal flow in natural co-ordinate can be written as

$$\frac{dv}{dt}\vec{t} + \frac{V^2}{R}\hat{n} = -\frac{1}{\rho}(\hat{t}\frac{\partial p}{\partial s} + \hat{n}\frac{\partial p}{\partial n}) - fv\hat{n} + (\hat{t}F_s + \hat{n}Fn)$$

Component wise
$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + F_s$$

$$\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fv + F_n$$

These are equation of motion in natural co-ordinate.

Gradient flow:

While discussing the geostrophic flow it was assumed that isobars are straight lines. But in reality isobars are curved lines.
Gradient flow is horizontal frictionless and isobaric flow i.e., parallel to isobars.
Since the flow is isobaric hence, $\frac{\partial p}{\partial s} = 0.$ so, v = constant. And,

$$\frac{V^2}{R} + \frac{1}{\rho}\frac{\partial p}{\partial n} + fv = 0$$

Thus gradient flow results from a three-way balance among the centrifugal force, coriolis force and pressure gradient force. This balances known as **gradient balance**.

Resultant between coriolis force and pressure gradient force gives rise to the necessary centripetal force required to maintain a curved flow, which is again equal and opposite to centrifugal force.
If the flow is straight, then $R \to \infty$, hence we have $\frac{1}{\rho}\frac{\partial p}{\partial n} + fv = 0$, the **geostrophic balance.**

Thus geostrophic balance is a special case of gradient balance for straight flow.
Wind obtained from gradient balance is known as **gradient wind.**
In the above gradient wind equation, $R \,\&\, \frac{\partial p}{\partial n}$ may have different sign. Hence we shall discuss all possible gradient flow for different combinations of sign of them.

Now from the gradient wind equation, V can be obtained as,

$$V = \frac{-fR \pm \sqrt{f^2R^2 - 4\frac{R}{\rho}\frac{\partial p}{\partial n}}}{2}$$

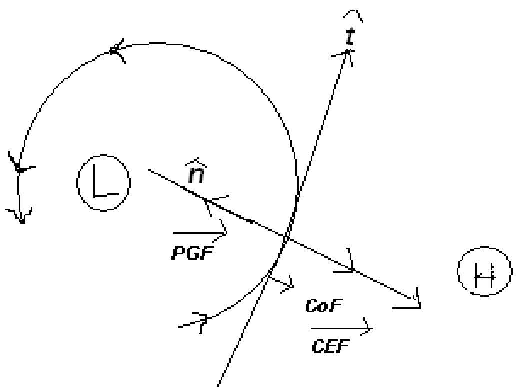
Case I: $R \& \frac{\partial p}{\partial n}$ both positive. Then the quantity inside the square root of expression for V is less than f^2R^2 and hence the square root of the quantity is less than fR. Hence both the positive and negative roots give negative root, which is not physically possible. Hence such gradient flow with $R \& \frac{\partial p}{\partial n}$ positive does not exist.

Case II: R positive and $\frac{\partial p}{\partial n}$ negative

Let $C = -\frac{\partial p}{\partial n}, hence$ C>0

Now, $V = \dfrac{-fR \pm \sqrt{f^2R^2 + 4\frac{RC}{\rho}}}{2}$

Clearly the quantity inside the square root is greater than f^2R^2. Hence the positive root only physically possible.
Possible gradient flow and the balance of force have been shown in the adjoining figure.



Clearly it is a cyclonic flow around a low. This flow is known as regular low. In this case $\overrightarrow{PGF}$ acts towards the center of curvature, where as $\overrightarrow{CoF} \& \overrightarrow{CFF}$ both are away from center of curvature. Hence in this case

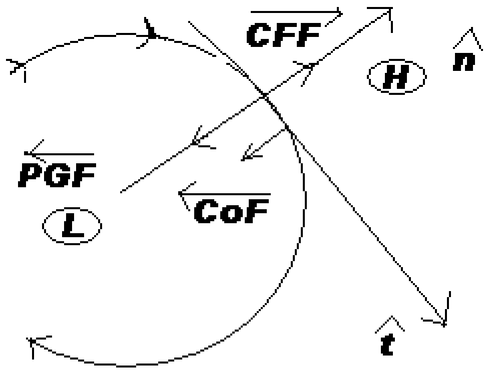
$$\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV$$

$$= \frac{C}{\rho} - fV$$

Case III : R negative and $\frac{\partial p}{\partial n} > 0$

Put S= -R>0,

Hence, $V = \dfrac{fS \pm \sqrt{f^2 S^2 + 4\frac{S}{\rho}\frac{\partial p}{\partial n}}}{2}$



Clearly the square root is more than fS and hence only positive root is physically possible. Possible gradient flow has been shown in the adjoining figure. It is an anticyclonic flow around a low, known as anomalous low.

Here both $\overrightarrow{PGF}$ & $\overrightarrow{CoF}$ are towards the centre of curvature. Where as $\overrightarrow{CFF}$ is away from the center of curvature.

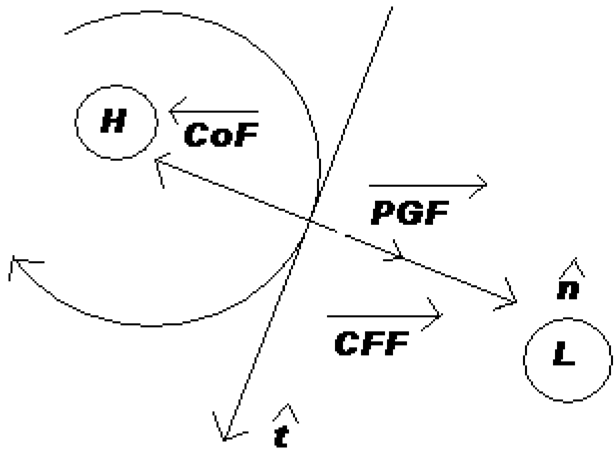
Hence in this case the magnitude of $\overrightarrow{CFF}$ is the addition of that of $\overrightarrow{PGF}$ & $\overrightarrow{CoF}$, where as in case of a regular low, magnitude of $\overrightarrow{CFF}$ is obtained by subtracting that of $\overrightarrow{CoF}$ from that of $\overrightarrow{PGF}$.

Hence for a given magnitude of radius of curvature and pressure gradient, gradient wind is stronger in an anomalous low than in a regular low.

Case IV: Both R & $\frac{\partial p}{\partial n}$ are negative.

Let $S = -R \rangle 0$ & $C = -\frac{\partial p}{\partial n} \rangle 0$

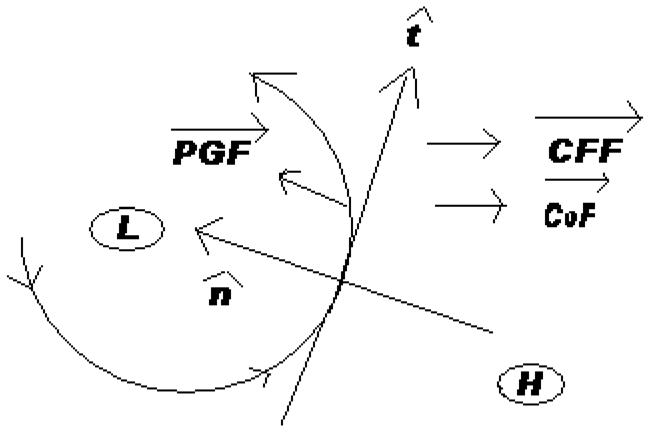
Hence, $V = \dfrac{fS \pm \sqrt{f^2 S^2 - 4\frac{SC}{\rho}}}{2}$



As, the square root is less than fS, hence both roots give physically possible solutions. For the positive root $V \rangle \frac{fS}{2} = -\frac{fR}{2}$ and the corresponding flow is known as anomalous high and for the negative root, $V \langle \frac{fS}{2} = -\frac{fR}{2}$, the corresponding flow is regular high. However in both cases flow is an anticyclonic flow around a high, which is shown in the adjoining figure.
We can see that $\overrightarrow{PGF}$ is very less for such flow.

Sub Geostrophic and super Geostrophic gradient wind: If the speed of gradient wind is more than that of Geostrophic wind, then the gradient wind is called super-geostrophic and otherwise if it is less than that of Geostrophic wind, then it is called sub-geostrophic.

From the adjoining figure it can be shown that the Cyclonic gradient flow is sub-geostrophic. We know that to execute any curvilinear flow, one requires centripetal acceleration.



One thing should be clear here, that the imbalance between the $\overrightarrow{PGF}$ & $\overrightarrow{CoF}$ only has given rise to the centripetal acceleration. The net resultant between these two forces is towards the direction of unit normal vector $\hat{n}$. Accordingly the ageostrophic wind is normal to $\hat{n}$ and to its left i.e.; opposite to the direction of horizontal wind. Hence the gradient wind, which is a resultant of geostrophic & ageostrophic wind, must be lesser than the geostrophic wind. Hence cyclonic gradient wind subgeostrophic.

Following similar argument one can offer a physical explanation for anticyclonic gradient wind to be super geostrophic.

It can be shown that gradient wind for anticyclonic flow is more than Geostrophic wind but less than four times the Geostrophic wind, i.e., $V_g < V_{gr} < 2V_g$.

Some special types of gradient balance/flow:

When we consider straight flow, $R \to \infty$ and gradient wind equation transforms into, $0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} - fV$, which is Geostrophic flow. In case of jet stream, curvature of the flow may not be significant enough for presence of centrifugal force.

When Coriolis force is very week, then gradient balance becomes the balance between the pressure gradient and centrifugal force, i.e.,

$\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n}$. This balance is known as cyclostrophic balance. Cyclones, anticyclones etc are suitable examples of cyclostrophic flow.

When pressure gradient is very weak, then the gradient balance becomes the balance between the coriolis force and centrifugal force, i.e., $\frac{V^2}{R} = -fV$. This balance is known as inertial balance. It can be shown that inertial flow is always anticyclonic. Inertial waves (easterly waves) at low latitudes are suitable examples for inertial flow.

Chapter-5
Vertical variation of wind

Vertical shear of geostrophic wind refers to rate of change of geostrophic wind vector with height, mathematically it is expressed as $\frac{\partial \vec{V}_g}{\partial z}$. Now we shall first try to understand how horizontal temperature gradient can lead to vertical shear of geostrophic wind.

We know that in isobaric co-ordinate, the equations for geostrophic wind are given by

$$u_g = -\frac{g}{f}\left(\frac{\partial z}{\partial y}\right)_p \text{ and } v_g = \frac{g}{f}\left(\frac{\partial z}{\partial x}\right)_p \text{, i.e., } \vec{V}_g = \frac{g}{f}\hat{k}\times\vec{\nabla}_p z \text{ ....(1)}$$

Thus, geostrophic wind at any isobaric level is directly proportional to the slope of that level.

Now let us consider two isobaric surfaces, which are having same slope initially. Now consider the layer between these two isobaric surfaces. Now we create horizontal temperature gradient in this layer by heating one part of the layer and cooling other part. This will cause a difference in the slopes of these two isobaric surfaces. This in tern will cause a difference in the geostrophic wind at these two levels, i.e., will cause a vertical shear of geostrophic wind. Thus horizontal temperature can lead to vertical shear of geostrophic wind.

Now we are in a position to give the definition of thermal wind. Thermal wind is defined as the vectorial difference between upper level geostrophic wind and lower level geostrophic wind. From the above discussion it follows that for thermal wind to exist there must be a horizontal temperature gradient. Thus we see that thermal wind owes to a thermal effect Viz., horizontal temperature gradient.

Equation of thermal wind : Let $\vec{V}_{gl}$ and $\vec{V}_{gu}$ are respectively the lower level and upper level geostropic wind at pressure levels P_l and P_u .Then, $\vec{V}_{gl}$ and $\vec{V}_{gu}$ are given by

$\vec{V}_{gl} = \frac{g}{f}\hat{k}\times\vec{\nabla}_p z_l$ and $\vec{V}_{gu} = \frac{g}{f}\hat{k}\times\vec{\nabla}_p z_u$, where $z_l(x,y)$ and $z_u(x,y)$ are geopotential height at different points at the above two levels. Then thermal wind $\vec{V}_T$, in the layer between pressure levels P_l and P_u is given by

$$\vec{V}_T = \vec{V}_{gu} - \vec{V}_{gl} = \frac{g}{f}\hat{k}\times\vec{\nabla}_p (z_u - z_l) \text{ ....(2)}$$

Again from Hypsometric equation we know that

$$z_u - z_l = \frac{R}{g}\ln\left(\frac{p_l}{p_u}\right)\hat{k}\times\vec{\nabla}_p < T > \text{....(3), where, <T> is given by,}$$

$$< T > = \frac{\int_{p_l}^{p_u} T d(\ln p)}{\int_{p_l}^{p_u} d(\ln p)} \text{ ....(4)}$$

Hence using (3) in (2) we have,

$$\vec{V}_T = \frac{R}{f}\ln\left(\frac{p_l}{p_u}\right)\hat{k}\times\vec{\nabla}_p < T > \textbf{....(5)}$$

Properties of thermal wind:

i. Thermal wind is not a real wind, it's a concept only.
ii. It refers to a layer not a level.
iii. It exists as long as there exists horizontal temperature gradient.
iv. It is parallel to mean isotherms in a layer, keeping colder side to the left (In northern hemisphere)
v. It is also parallel to mean thickness lines of the layer, keeping lower thickness to the left (in Northern hemisphere)

Concept of barotropic and baroclinic atmosphere:

An atmosphere is said to be barotropic if the density is a function of pressure only. Hence, $\rho = f(p)$. This functional relation along with the equation of state $p = \rho RT$ gives $T = h(p)$.

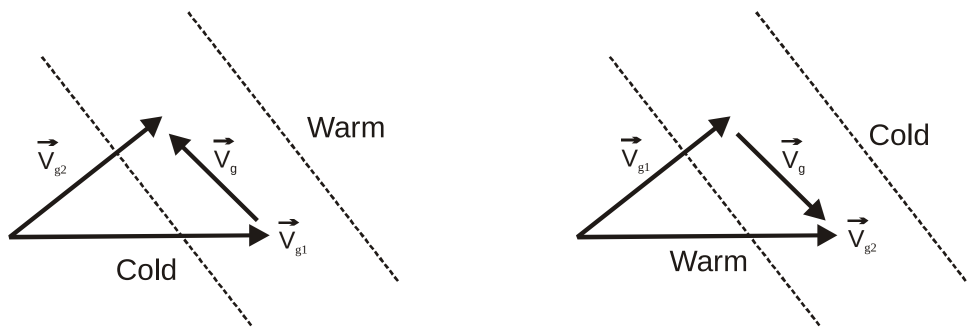
Thus, $\vec{\nabla}T = h'(p)\vec{\nabla}p$

$\Rightarrow \vec{\nabla}T \ II \ \vec{\nabla}p \Rightarrow$ isotherms are parallel to isobars, i.e., there is no change in T along isobars, i.e., horizontal temperature gradient on an isobaric surface is zero for a barotropic atmosphere. Thus in a barotropic atmosphere, the geostrophic wind does not change with height and thermal wind is zero.

Otherwise if density is not a function of pressure only, then it is called a baroclinic atmosphere. So, thermal wind exists only in a baroclinic atmosphere.

Backing and veering of Geostrophic wind:

The change in the geospheric wind with height is also an indicator of thermal advection. The same is shown graphically.



Cold air advection
Backing

Warm air advection
Vearing

Backing refers to anti-clockwise turn of geostrophic wind with height, where as veering refers to clockwise turn of geostrophic wind with height. The above two figures explain warm (cold) air advection associated with veering (backing).

Application of the thermal wind concept:

i. Using thermal wind concept one can show that cold (warm) core low (high) intensifies with height.
ii. Using thermal wind concept one can, explain the existence of sub tropical westerly jet in winter and tropical easterly jet in southwest monsoon season over India.
iii. Using thermal wind concept one can explain warm (cold) air advection associated with veering (backing).

Chapter-6
Kinematics of wind and pressure field

Streamline: Stream lines are curves, the tangent to each point of which is represented by horizontal wind vector at that point. Thus if at an arbitrary point (x, y), $u(x, y)$ and $v(x, y)$ are horizontal components of wind, then we have,

$\frac{dy}{dx} = \frac{v(x, y)}{u(x, y)}$. This is the differential equation of stream line. Stream lines give the picture of instantaneous motion.

Trajectory: It is the actual path traced by an air parcel during a finite interval of time. Thus if $s(x, y, t)$ is the displacement at time 't' and V(x,y,t) is the speed, then $\frac{ds}{dt} = V(x, y, t)$. It gives the picture of total motion during an interval of time.

Relation between streamlines and trajectories: It is expressed by an equation, known as, Blaton's equation, which is given by

$\frac{\partial \beta}{\partial t} = V(\kappa_T - \kappa_S)$; where,

β = Angle of turning of wind

$V =$ Wind speed

$\kappa_T =$ Curvature of trajectory

$\kappa_S =$ Curvature of streamline.

Under steady state condition, left hand side of above equation vanishes and hence stream line and trajectory coincides.

Stream line pattern for different wind field: It can be shown that horizontal motion consists of pure translation, pure divergence, pure rotation and pure deformation.

Equation of stream lines for pure translation is given by: $y = mx + c$; $m = \frac{v(x_{0,} y_0)}{u(x_0, y_0)}$,

(x_0, y_0) is the centre and c is a parameter. Thus for pure translational motion stream lines are family of parallel straight lines with slope 'm'.

For pure divergence, stream lines are family of straight lines passing through centre (origin), given by the equation: $y = kx$; k being a parameter.

For pure rotational motion equation of stream lines is given by:

$x^2 + y^2 = a^2$; 'a' being a parameter. Thus for pure rotational motion, stream lines are family of concentric circles. Similarly it can be shown that for pure deformation, stream line patterns are either family of hyperbola $(x^2 - y^2 = contant)$ with same foci or family of rectangular hyperbola $(xy = contant)$.

Centre of low, high, trough of low pressure , ridge of high pressure and COL

An arbitrary point, (x_0, y_0) is said to be a centre of low pressure/ high pressure if the pressure field $p(x, y)$ has a local minima/ maxima at this point, i.e., if

Conditions for low pressure centre	Conditions for high pressure centre
$\left(\frac{\partial p}{\partial x}\right)_{(x_0,y_0)} = 0$	$\left(\frac{\partial p}{\partial x}\right)_{(x_0,y_0)} = 0$
$\left(\frac{\partial p}{\partial y}\right)_{(x_0,y_0)} = 0$	$\left(\frac{\partial p}{\partial y}\right)_{(x_0,y_0)} = 0$
$\left(\frac{\partial^2 p}{\partial x^2}\right)_{(x_0,y_0)} > 0$ and $\left(\frac{\partial^2 p}{\partial y^2}\right)_{(x_0,y_0)} > 0$	$\left(\frac{\partial^2 p}{\partial x^2}\right)_{(x_0,y_0)} < 0$ and $\left(\frac{\partial^2 p}{\partial y^2}\right)_{(x_0,y_0)} < 0$

Conditions stated at first two rows states that the centre of high/low is a stationary point and condition stated in last row implies that around a centre of low (or high), the isobars turn in anti clock wise (or clock wise) direction.

Before defining trough of low pressure or ridge of high pressure, we should first understand about axis of symmetry. An arbitrary axis, say y-axis, is said to be an axis of symmetry for a quantity, 'S', if each point on this axis is either a local minima or a local maxima for 'S', i.e., if either $\left(\frac{\partial S}{\partial x}\right) = 0$ and $\left(\frac{\partial^2 S}{\partial x^2}\right) > 0$ or $\left(\frac{\partial S}{\partial x}\right) = 0$ and $\left(\frac{\partial^2 S}{\partial x^2}\right) < 0$.

Trough of low pressure is a line of symmetry, each point of which is a local minima for the pressure field $p(x, y)$ i.e., at each point on the trough $\left(\frac{\partial p}{\partial x}\right) = 0$ and $\left(\frac{\partial^2 p}{\partial x^2}\right) > 0$.

Similarly, ridge of high pressure is a line of symmetry, each point of which is a local maxima for the pressure field $p(x, y)$ i.e., at each point on the ridge $\left(\frac{\partial p}{\partial x}\right) = 0$ and $\left(\frac{\partial^2 p}{\partial x^2}\right) < 0$.

From the above, it follows that around a trough/ridge, isobars (isolines) turn in anti clock wise (or clock wise) direction.

Centre of COL is the point of intersection between one trough and ridge. If the trough is oriented along y-axis and ridge is oriented along x-axis, then at the COL, we have

$\left(\frac{\partial p}{\partial x}\right) = 0$ and $\left(\frac{\partial^2 p}{\partial x^2}\right) > 0$, $\left(\frac{\partial p}{\partial y}\right) = 0$ and $\left(\frac{\partial^2 p}{\partial y^2}\right) < 0$.

Chapter-7
Divergence & vorticity

Before discussing about conservation laws for mass and water vapour, one must first under stand about the concept of divergence. So in the next section, we shall discuss about divergence.

Divergence: Divergence of an arbitrary vector field $\vec{B}$ is its measure of outflow. It is defined as $\vec{\nabla}.\vec{B}$. In Meteorology, by the term divergence, we mean divergence of wind vector, $\vec{V}$. Thus in meteorology divergence is mathematically expressed as $\vec{\nabla}.\vec{V}$. If the value of this quantity is positive then we say that divergence is taking place and if the value of this quantity is negative then we say that convergence is taking place We may talk of 2 dimensional as well as 3 dimensional divergence, which are occasionally denoted as $\vec{\nabla}_2.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and $\vec{\nabla}_3.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$. It can be shown that 2 dimensional divergence is equal to fractional rate of change of area, i.e., the time rate at which a unit area is expanding or contracting. It can be mathematically expressed as $\vec{\nabla}_2.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{A}\frac{dA}{dt}$, A being the area. Similarly, it can be shown that 3 dimensional divergence is equal to fractional rate of change of volume, i.e., the time rate at which a unit volume is expanding or contracting. It can be mathematically expressed as $\vec{\nabla}_3.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{1}{\sigma}\frac{d\sigma}{dt}$, σ being the volume. Physically one can visualize that expansion/contraction of an area or volume is possible only by outflow (divergence)/inflow(convergence) of air.

In natural co-ordinate, 2-dimentional divergence (conventionally called as horizontal divergence) is given by $\vec{\nabla}_2.\vec{V} = \frac{\partial V}{\partial s} + V\kappa_n$, V is the wind speed and κ_n is the orthogonal curvature of the stream line. Orthogonal curvature of the stream line is nothing but the curvature of a curve which is orthogonal to the stream line. A curve, C_1 is said to be orthogonal to another curve, say, C_2 at a point, if the tangents at this point to both the curves are perpendicular to each other.

From the above expression of horizontal divergence, it follows that down wind increase/decrease of wind speed gives rise to divergence/convergence. It also follows that a positive(cyclonic)/negative (anti-cyclonic) orthogonal curvature gives rise to divergence/convergence.

VORTICITY:

Vorticity is a micro scale measure of rotation. It is a vector quantity. Direction of this vector quantity is determined by the direction of movement of a fluid, when it is being rotated in a plane. Observation shows that when a fluid is being rotated in a plane, then there is a tendency of fluid movement in a direction normal to the plane of rotation (towards outward normal if rotated anti clockwise or towards inward normal if rotated clockwise). Thus due to rotation in the XY plane (Horizontal plane) fluid tends to move in the $\hat{k}$ direction (i.e. vertical), due to rotation in the YZ plane (meridional vertical plane)fluid tends to move in the $\hat{i}$ direction (East West) and due to rotation in ZX plane (zonal vertical plane) fluid tends to move in the $\hat{j}$ direction (N-S).

Thus vorticity has three components. Mathematically it is expressed as

$\vec{\nabla}\times\vec{V} = \hat{i}\xi + \hat{j}\eta + \hat{k}\zeta$ where, $\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}; \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}; \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

In Meteorology, we are concerned about weather, which is due mainly to vertical motion and also only the rotation in the horizontal plane can give rise to vertical motion. So, in Meteorology, by the term vorticity, only the $\hat{k}$ component of the vorticity vector is understood. Hence, throughout our study only $\hat{k}$ component is implied by vorticity.

Thus, hence forth, vorticity $= \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. It is known as relative vorticity. It is solely due to motion of an object. But rotation of earth itself gives rise to some vorticity, even to a stationary object. Vorticity, which is solely due to rotation of earth, is known as planetary vorticity. It can be shown that, planetary vorticity at a latitude φ = f= 2ΩSinφ. Planetary vorticity and relative vorticity, combined together, is known as absolute vorticity and is denoted by $\varsigma_a = \varsigma + f$.

In natural co-ordinate, relative vorticity is expressed as

$\zeta = V\kappa_s - \frac{\partial V}{\partial n}$, where, κ_s is the stream line curvature and $\frac{\partial V}{\partial n}$ is across the streamline wind shear. First term on right hand side of the above expression is known as curvature vorticity and the second term (including sign) is known as shear vorticity. From first term it follows that at an arbitrary point, vorticity will be cyclonic/anti-cyclonic if the curvature of the stream line at that point is cyclonic / anti-cyclonic. Similarly from the second term, it follows that if wind speed decreases/increases to the left across the stream line, then there will be cyclonic/ anti cyclonic vorticity.

Due to effect of Coriolis force, the out flowing (inflowing) stream lines (which are straight lines) in a divergent/convergent field, are deflected to the right and acquire anti-cyclonic curvature/cyclonic curvature. That’s why divergence is associated with anticyclonic vorticity and convergence is associated with cyclonic vorticity.

It can be shown that for synoptic and larger scale motion, change in vorticity is mainly due to large scale divergence/convergence.

Chapter – 8
Conservation of mass

Atmospheric motion is governed by some conservation laws, viz., the conservation of momentum, conservation of mass, conservation of energy and conservation of moisture. Conservation of momentum has already been discussed in the equation of motion.

Now, we shall discuss the conservation of mass, which says that mass remains conserved. In this context of establishing the laws of conservation, it is worth to mention that conservation laws can be established following two dynamical approaches,viz, Eulerian approach and Lagrangian approach.

In both approach an air parcel is considered. In the former approach air parcel is considered stationary while fluid is allowed to pass through it and in the later approach the motion of the parcel along with flow in considered.

Mathematical equation expressing the law of conservation of mass is known as mass continuity equation.

Following either of the above mentioned approach, we can arrive at following two forms of mass continuity equation, viz., the mass divergence form

$\frac{\partial \rho_0}{\partial t} = -\vec{\nabla}.(\rho_0 \vec{v})$, which states that in-situ/local change in mass at a point is solely due to mass convergence/divergence at that point and the velocity divergence form

$\frac{1}{\rho_0}\frac{d\rho_o}{dt} = -\vec{\nabla}.\vec{v}$, which states that total change of mass at a point is solely due to convergence/divergence of wind.

Continuity equation in pressure co-ordinate is given by

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$, where $\omega = \frac{dp}{dt}$, the vertical velocity in pressure co-ordinate and its unit is hPa/Sec. It can be shown that $\omega \approx -g\rho w$, where w is the vertical velocity in m/Sec. Thus one can see that ω is positive for sinking motion and negative for rising motion.

Is 'P' as a vertical co-ordinate is superior than 'Z'? If so then how?

Pressure as a vertical co-ordinate is superior than Z, due to following reasons:

- Density ‘ ρ ‘ does not appear explicitly in the governing equations in p-coordinate.
- The above is beneficial due to the fact that ‘ ρ ’ is not an observed field, rather it is a derived field. Use of it explicitly requires extra computations.
- As ‘ ρ ’ is not appearing explicitly, rarefication and compression are eliminated resulting into completely removal of sound wave-a meteorological noise.

Moisture Continuity Equation:

If q is the specific humidity of water vapour in air and if ρ is the density of air, then density of water vapour is ρq. Thus applying mass continuity equation to water vapour mass, we get

$$\frac{\partial(q\rho)}{\partial t} = -\vec{\nabla}.(\rho q\vec{v}) \Rightarrow \frac{1}{q}\frac{\partial q}{\partial t} + \frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{\rho}\vec{\nabla}.(\rho\vec{v}) - \frac{1}{q}\vec{v}.\vec{\nabla}q$$

Hence, using mass continuity equation, viz., $\frac{\partial \rho}{\partial t} = -\vec{\nabla}.(\rho\vec{v})$, we obtain

$\frac{\partial q}{\partial t} + \vec{v}.\vec{\nabla}q = 0 \Rightarrow \frac{dq}{dt} = 0$. This is known as moisture continuity equation.

Dines compensation principle: If the mass conservation law is applied to an air column, then one can find that convergence throughout entire column or divergence throughout entire column, would lead to a net increase or net decrease of mass in the column, both of which contradict law of conservation of mass. Thus only convergence or divergence throughout an air column is not possible. Convergence in some layers must be compensated by divergence in some other layers. This is known as Dines compensation principle.

This principle leads us to a level, theoretically at which there shouldn’t be any divergence or convergence. This is known as level of non-divergence. For all operational purpose, 500hPa level is considered to be the level of non divergence. If the observed divergence at this level is more than its climatological normal value, then it’s an indication of enhancement of low-level convergence. This is an important forecasting tool.

Application of continuity equation: Mass continuity equation in isobaric co-ordinate can be used to find out vertical velocity at any level. Integrating this equation between two adjacent pressure levels, say, P_L and P_U, we obtain,

$\omega(P_U) = \omega(P_L) + \langle D \rangle \Delta P$ **;** Where

$\Delta P = P_L - P_U$ **,**

$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$,

$\langle\ \rangle$ Refers to mean of the values at two levels and

$\omega(P_L), \omega(P_U)$ are vertical velocities in hPa/Sec at two levels.

Chapter-9
Planetary Boundary Layer (PBL)

A Brief essay on PBL:

PBL is the lower most portion of the atmosphere, adjacent to the earth's surface, where maximum interaction between the Earth surface and the atmosphere takes place and thereby maximum exchange of Physical properties like momentum, heat, moisture etc., are taking place.

Exchange of physical properties in the PBL is done by turbulent motion, which is a characteristic feature of PBL. Turbulent motion may be convectively generated or it may be mechanically generated.

If the lapse rate near the surface is super adiabatic, then PBL becomes positively Buoyant, which is favorable for convective motion. In such case PBL is characterized by convective turbulence. Generally over tropical oceanic region with high sea surface temperature this convective turbulence occurs. If the lapse rate near the surface is sub adiabatic then the PBL is negatively buoyant and it is not favourable for convective turbulence. But in such case, if there is vertical shear of horizontal wind, then Vortex (cyclonic or anti cyclonic) sets in, in the vertical planes in PBL. This vortex motion causes turbulence in the PBL, known as mechanical turbulence.

If the PBL is positively buoyant as well as, if vertical shear of the horizontal wind exists, then both convective and Mechanical turbulence exits in the PBL.

Now, we consider a typical situation, when PBL is stably stratified and there exists vertical shear of mean horizontal wind. In such situation convective turbulence is suppressed but mechanical turbulence is enhanced. In this situation it is difficult to say whether their combined effect is to suppress turbulence or to sustain turbulence. This situation is tackled using a non-dimentional number, which is defined as the ratio between, static stability and square of vertical shear of horizontal wind, i.e.,

Richardson number = $R_f = \dfrac{\dfrac{g}{\theta}\dfrac{d\theta}{dz}}{\left[\dfrac{\partial u}{\partial z}\right]^2 + \left[\dfrac{\partial v}{\partial z}\right]^2}$ Empirically it has been found that, R_f must be less than 0.25 to sustain the turbulence. Thus R_f should be less than ¼ to maintain turbulence in a stably stratified PBL by wind shear.

The depth of the PBL is determined by the maximum vertical extent to which the turbulent motion exists in PBL. On average it varies from few cms to few kms. In case of thunderstorms PBL may extend up to tropopause.

Generally at a place on a day depth of PBL is maximum at noon and in a season it is maximum during summer.

Division of the PBL into different sub layers:

The PBL may be sub divided into three different sections, viz viscous sub layer, the surface layer and the Ekmann layer or entrainment layer or the transition layer.

Viscous layer is defined as the layer near the ground, where the transfer of physical quantities by molecular motions becomes important. In this layer frictional force is comparable with PGF.

The surface layer extends from z = z_0 (roughness length) to $z = z_s$ with z_s, the top of the surface layer, usually varying from 10 m to 100 m. In this layer sub grid scale fluxes of momentum (eddy stress) and frictional forces are comparable with PGF.

The last layer is the Ekmann layer is defined to occur from z_s to z_i, which ranges from 100 m or so to several kilometers or more. Above the surface layer, the mean wind changes direction with height and approaches to free stream velocity at the height z as the sub grid scale fluxes decrease in magnitude. In this layer both the COF and Eddy stress are comparable with PGF.